

# Conformal Blocks and Negativity at Large Central Charge

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We consider entanglement negativity for two disjoint intervals in 1+1 dimensional CFT in the limit of large central charge. As the two intervals get close, the leading behavior of negativity is given by the logarithm of the conformal block where a set of approximately null descendants appears in the intermediate channel. We compute this quantity numerically and compare with existing analytic methods which provide perturbative expansion in powers of the cross-ratio.

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## 1. Introduction and summary

Entanglement entropy has recently appeared in a variety of contexts ranging from AdS/CFT [1-2] to entropic derivations of the c-theorems [3-5]. Some reviews of the subject include [6-9]. An alternative measure of entanglement, called entanglement negativity, has been introduced in [10] and further discussed in [11] and [12](see also [13-17]). The entanglement negativity appears to be a good entanglement measure for systems in a mixed state. It was additionally shown that in 1+1 dimensional conformal field theories (CFTs) the value of the entanglement negativity for a single interval is proportional to the central charge of the theory, similarly to entanglement entropy. Hence, entanglement negativity might be useful for defining a proper measure of degrees of freedom, something that entanglement entropy is having some issues with, especially in four and higher space-time dimensions [5].

The holographic prescription for computing entanglement entropy [1-2] has given us a simple and efficient computational technique. A derivation based on the replica trick was provided in [18]. The key property was the  $n \rightarrow 1$  limit of the replica index: the contribution proportional to  $(n-1)$  is localized on minimal surfaces in AdS. Entanglement negativity also involves  $n \rightarrow 1$ , but is supposed to be finite in this limit, thanks to the non-trivial analytical continuation from even  $n$  (to be reviewed below). Hence, it is non clear at present whether a nice holographic formula for entanglement negativity exists. (See [19] for a recent discussion.)

In two space-time dimensions conformal symmetry is much more powerful than in higher dimensions, and one can recover some of the holographic results for entanglement entropy by going to the limit of large central charge and making some mild assumption about the spectrum of the operators and the OPE coefficients [20] (see also [21-24] and for related work [25-29]). One may ask whether a similar approach can be useful in the computation of entanglement negativity, and if the holographic prescription can be inferred from it.

In this paper we attempt to answer the first question by computing entanglement negativity for two disjoint intervals in 1+1 dimensional CFTs in the limit of large central charge  $c$ . Technically we need to compute a four-point function of twist operators in an arbitrary CFT. The difference from the entanglement entropy computation arises from the fact that in the limit of close intervals the two operators that come together are not the twist and its inverse, but rather two identical twist operators. As a result, the leading

contribution in this limit does not come from the identity, but from a non-trivial twist operator. As in [20], we assume that the computation of the conformal block, associated with this operator is sufficient to give the full result for the negativity (unlike [20], we do not have a non-trivial holographic check of this assumption).

Generally there exists a number of ways to compute conformal blocks at large  $c$  as an expansion in powers of the cross-ratio. However the conformal block that appears in the result for negativity is particularly non-trivial: the internal operator has a descendant which is null in the limit of infinite central charge. This shows up as a divergence in the next to next to leading term in the expansion. In situations like this one generally expects a series of divergent terms which can be resummed. This is technically a hard problem, so instead we reformulate the problem in terms of a auxiliary differential equation with fixed monodromy, and solve it numerically. Interestingly, we find two solutions; we pick up the dominant one, while the other is exponentially suppressed in the limit of large central charge.

The rest of the paper is organized as follows. In the next Section we review the definition of entanglement negativity and how it can be computed in a 1+1 dimensional CFT using the replica trick. In Section 3 we describe our result. We discuss it in Section 4. In the Appendix we explore some analytic techniques for computing the conformal block as an expansion in powers of the cross-ratio.

## 2. Negativity in a 1+1 dimensional CFT: a review

In this Section we briefly review the results of [11] and [12]. We start by recalling the definition of negativity [10], used as a measure of entanglement between two quantum subsystems  $A_1$  and  $A_2$  whose Hilbert spaces are denoted by  $\mathcal{H}_1$  and  $\mathcal{H}_2$  below. Denote by  $|e_i^{(1)}\rangle$  and  $|e_j^{(2)}\rangle$  the bases of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  respectively. The partial transpose of the density matrix  $\rho$  in  $\mathcal{H}_1 \cup \mathcal{H}_2$  is defined as

$$\langle e_i^{(1)} e_j^{(2)} | \rho^{T_2} | e_k^{(1)} e_l^{(2)} \rangle = \langle e_i^{(1)} e_l^{(2)} | \rho | e_k^{(1)} e_j^{(2)} \rangle. \quad (2.1)$$

Then the entanglement negativity is simply defined as

$$\mathcal{E} \equiv \ln \text{tr} |\rho^{T_2}|, \quad (2.2)$$

where  $\text{tr}|\rho^{T_2}|$  denotes the sum of the absolute values of the eigenvalues of  $\rho^{T_2}$ . The transposed matrix is normalized,  $\text{tr}(\rho^{T_2}) = 1$ , but it is not necessarily a density matrix, since it can have negative eigenvalues that are counted by the negativity, hence the name.

As explained in [11], one can compute the value of  $\mathcal{E}$  using the replica trick. To see how, it is useful to express the trace of  $\rho^{T_2}$  in terms of its eigenvalues  $\lambda_i$

$$\text{tr}|\rho^{T_2}| = \sum_i |\lambda_i| = \sum_{\lambda_i > 0} \lambda_i + \sum_{\lambda_i < 0} |\lambda_i|. \quad (2.3)$$

and subsequently consider the trace of integer powers of  $\rho^{T_2}$ . It is evident that the dependence of  $\text{tr}(\rho^{T_2})^n$  on  $|\lambda_i|$  will differ for odd and even integer values of  $n$ , *i.e.*,

$$\text{tr}(\rho^{T_2})^n = \sum_i \lambda_i^n = \begin{cases} \sum_{\lambda_i > 0} \lambda_i^n + \sum_{\lambda_i < 0} |\lambda_i|^n, & n : \text{even} \\ \sum_{\lambda_i > 0} \lambda_i^n - \sum_{\lambda_i < 0} |\lambda_i|^n, & n : \text{odd} \end{cases}. \quad (2.4)$$

The authors of [11] observed that if one formally sets  $n = 1$  in the upper line of eq. (2.4), one obtains exactly eq.(2.3). This is not true for the case of odd integer  $n$ , where the lower line of eq.(2.4) simply yields the normalization condition  $\text{tr}\rho^{T_2} = 1$ . Disregarding issues with the existence of a unique analytic continuation, entanglement negativity can thus be obtained by analytically continuing the *even* sequence of integer numbers  $n$  and then taking the limit  $n \rightarrow 1$ ,

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \ln \text{tr}(\rho^{T_2})^{n_e}, \quad (2.5)$$

where, following the conventions of [11] we defined  $n_e \equiv 2m$  for integer  $m$ .

A consistency check for the replica trick method is performed in [11] where  $\mathcal{E}$  is evaluated for the case of a bipartite system  $\mathcal{H}_A \times \mathcal{H}_B$  in a pure state  $|\Psi\rangle$  and the result is shown to agree with [10], *i.e.*, entanglement negativity is equal to the Renyi entropy of order 1/2

$$\mathcal{E} = 2 \ln [\text{tr}(\rho_A^{\frac{1}{2}})]. \quad (2.6)$$

The replica trick has been very useful in obtaining results for entanglement entropy in 2d CFTs, see for example [6] and references therein. The success was due to expressing integer powers of the trace of the reduced density matrix in terms of correlation functions of twist operators which are fixed by conformal invariance up to a few independent parameters. It turns out [11] that one can similarly express  $\text{tr}[(\rho^T)^n]$  in terms of twist operators

and thus compute  $\mathcal{E}$  in a few simple cases in  $1+1$  dimensional CFTs. However, the situation for entanglement negativity is slightly more complicated than that for entanglement entropy, due to the nature of the analytic continuation (from even  $n$  to  $n \rightarrow 1$  in (2.5)).

The simplest non-trivial configuration to consider involves two disjoint intervals,

$$A_1 = (z_1, z_2), \quad A_2 = (z_3, z_4), \quad z_2 < z_3 \quad . \quad (2.7)$$

The authors of [11] showed that in this case we can identify integer powers of the transpose of the reduced density matrix with the following four-point function of twist operators

$$\text{tr}(\rho_{A_2}^T)^n = \langle \mathcal{T}_n(z_1) \overline{\mathcal{T}}_n(z_2) \overline{\mathcal{T}}_n(z_3) \mathcal{T}_n(z_4) \rangle \quad (2.8)$$

where  $\mathcal{T}_n, \overline{\mathcal{T}}_n$  have conformal dimensions (see for example [30]):

$$h_{\mathcal{T}_n} = h_{\overline{\mathcal{T}}_n} = \frac{c}{24} \left( n - \frac{1}{n} \right) \equiv h \quad (2.9)$$

Following eq. (2.5) entanglement negativity can thus be obtained from the analytic continuation of the logarithm of this four-point function for even  $n$ ,

$$\mathcal{E} = \lim_{n_e \rightarrow 1} \ln [\langle \mathcal{T}_{n_e}(z_1) \overline{\mathcal{T}}_{n_e}(z_2) \overline{\mathcal{T}}_{n_e}(z_3) \mathcal{T}_{n_e}(z_4) \rangle] \quad (2.10)$$

Conformal invariance implies that entanglement negativity is a function of the cross-ratio,

$$x = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)} = \frac{\ell_1 \ell_2}{(\ell_1 - (z_3 - z_2))(\ell_2 + (z_3 - z_2))} \quad (2.11)$$

where we defined  $\ell_1$  and  $\ell_2$  to be the length of the intervals  $A_1$  and  $A_2$  respectively. An interesting limit involves two intervals that are taken to be very close to each other. In this adjacent interval limit,  $x \rightarrow 1$  and

$$\mathcal{E} \simeq -\frac{c}{4} \log(1 - x) \quad (2.12)$$

Another limit involves taking the two intervals far away from each other,  $x \rightarrow 0$ . As explained in [12], in this limit negativity is non-perturbatively small: all coefficients in front of powers of  $x$  vanish identically.

### 3. Entanglement negativity in the limit of large central charge

In this section, we will review the general characteristics of four point functions of primary operators in the limit of large central charge (for a detailed recent review see e.g. [31]). We will then proceed to study the four-point function through which entanglement negativity is defined (eq. (2.10)) and compute  $\mathcal{E}$  using some mild assumptions about the behavior of the ope coefficients and the spectrum of operators. The assumptions are exactly the ones used in [20] and which reproduced the holographic result for the entanglement entropy of disjoint intervals.

Consider the four point function of primary operators  $\langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4) \rangle$ . Conformal invariance allows us to set  $z_1 = 0, z_2 = x, z_3 = 1, z_4 = \infty$  and focus on  $\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_3(1)\mathcal{O}_4(\infty) \rangle$ . Moreover, it implies that any four point function of primary operators  $\mathcal{O}_i$  can be decomposed into conformal blocks

$$\langle \mathcal{O}_1(0)\mathcal{O}_2(x)\mathcal{O}_3(1)\mathcal{O}_4(\infty) \rangle = \sum_p a_p \mathcal{F}(c, h_p, h_i, x) \bar{\mathcal{F}}(c, \bar{h}_p, \bar{h}_i, \bar{x}), \quad (3.1)$$

where  $(h_i, \bar{h}_i)$  denote the conformal dimensions of the operators  $\mathcal{O}_i$  and the summation is over primary operators  $\mathcal{O}_p$  with conformal dimension  $(h_p, \bar{h}_p)$ . An analytic expression for  $\mathcal{F}(c, h_p, h_i, x)$  is known only for particular values of the parameters. In general, it is computed in the form of a series expansion in powers of  $x$  or  $(1-x)$  depending on whether (3.1) is a  $s$ - or  $t$ -channel decomposition into conformal blocks.

However, in the limit of large central charge  $c \gg 1$  and fixed  $\left(\frac{h_p}{c}, \frac{h_i}{c}\right)$ , the conformal blocks acquire a simple exponential form from [32-33]

$$\mathcal{F}(c, h_p, h_i, x) \sim \exp \left[ -\frac{c}{6} f \left( \frac{h_p}{c}, \frac{h_i}{c}, x \right) \right]. \quad (3.2)$$

The function  $f \left( \frac{h_p}{c}, \frac{h_i}{c}, x \right)$  is then determined by the monodromy properties of a second order ordinary differential equation,

$$\psi''(z) + T(z)\psi(z) = 0 \quad (3.3)$$

where

$$T(z) = \sum_{i=1}^{i=4} \left( \frac{6h_i}{c(z-z_i)^2} - \frac{c_i}{z-z_i} \right) \quad (3.4)$$

is related to the semiclassical, *i.e.*, large  $c$ , stress energy tensor through the Ward Identity  $\langle T(z)\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4) \rangle = \sum_{i=1}^4 \left( \frac{h_i}{(z-z_i)^2} + \frac{\partial_i}{z-z_i} \right) \langle \mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\mathcal{O}_4(z_4) \rangle$ .

The constants  $c_i$  in (3.3) are called accessory parameters. Requiring  $T(z)$  to behave like  $z^{-4}$  as  $z \rightarrow \infty$  leads to

$$\sum_i c_i = 0, \quad \sum_i c_i z_i - \frac{6h_i}{c} = 0, \quad \sum_i c_i z_i^2 - \frac{12h_i}{c} z_i = 0. \quad (3.5)$$

These equations determine three out of the four accessory coefficients so that, after setting  $(z_1, z_2, z_3, z_4) = (0, x, 1, \infty)$ ,  $T(z)$  becomes

$$T(z) = \frac{6h_1}{cz^2} + \frac{6h_2}{c(z-x)^2} + \frac{6h_3}{c(z-1)^2} + \frac{6(h_1 + h_2 + h_3 - h_4)}{cz(1-z)} - \frac{c_2 x(1-x)}{z(z-x)(1-z)}. \quad (3.6)$$

To specify  $c_2(x)$  one is then instructed to consider the trace of the monodromy matrix  $M$  of the solutions  $\psi_{1,2}(z)$  of (3.3) around a path enclosing points  $(0, x)$  for the  $s$ -channel or  $(x, 1)$  in the  $t$ -channel and require that

$$\text{tr} M = -2 \cos \pi \Lambda_p, \quad h_p = \frac{c}{24}(1 - \Lambda_p^2) \quad (3.7)$$

Finally, the semiclassical conformal block  $f(c, h_p, h_i, x)$  is related to  $c_2(x)$  via

$$\frac{\partial f}{\partial x} = c_2(x) \quad (3.8)$$

Assuming now that  $f(c, h_p, h_i, x)$  is known for all operators  $\mathcal{O}_p$ , one can compute the four point function (3.1) in the semiclassical limit via

$$\langle \mathcal{O}_1(0) \mathcal{O}_2(x) \mathcal{O}_3(1) \mathcal{O}_4(\infty) \rangle \simeq \sum_p a_p \exp \left[ -\frac{c}{6} f \left( \frac{h_p}{c}, \frac{h_i}{c}, x \right) - \frac{c}{6} \bar{f} \left( \frac{h_p}{c}, \frac{h_i}{c}, x \right) \right]. \quad (3.9)$$

It was argued in [20] that, under some mild assumptions, in the semiclassical regime the dominant contribution to (3.9) comes from the conformal block associated to operators  $\mathcal{O}_p$  of the lowest dimension  $\frac{h_p}{c}$ . In other words, (3.9) can be well approximated by the first term in the sum.

The analysis above was used in [20] to compute the entanglement entropy of two disjoint intervals in the limit of large central charge (recall that the entanglement entropy of two disjoint intervals is related to  $\langle \mathcal{T}_n(0) \bar{\mathcal{T}}_n(x) \mathcal{T}_n(1) \bar{\mathcal{T}}_n(\infty) \rangle$ ). Ref. [20] showed that the light operators to consider are the dimension zero operators which correspond to trivial monodromy according to (3.7). The result obtained precisely matched the one derived earlier from holography [2].

We would like to perform a similar analysis to compute the entanglement negativity for two disjoint intervals. As mentioned in the previous section, we should focus on the following four point function of twist operators

$$\langle \mathcal{T}_{n_e}(z_1) \overline{\mathcal{T}}_{n_e}(z_2) \overline{\mathcal{T}}_{n_e}(z_3) \mathcal{T}_{n_e}(z_4) \rangle \quad (3.10)$$

with conformal dimensions given by eq.(2.9). We shall again assume that the conformal block with the smallest conformal dimension provides the leading contribution to the sum (3.9). So the problem amounts to identifying the operator  $\mathcal{O}_p$  with the smallest dimension.

Let us consider the limiting cases  $x = 0$  and  $x = 1$ . For  $x \rightarrow 0$  the four point function of (3.10) leads to

$$\lim_{x \rightarrow 0} \langle \mathcal{T}_n(0) \overline{\mathcal{T}}_n(x) \overline{\mathcal{T}}_n(1) \mathcal{T}_n(\infty) \rangle = \langle \mathcal{I}(0) \overline{\mathcal{T}}_n(1) \mathcal{T}_n(\infty) \rangle = \langle \overline{\mathcal{T}}_n(1) \mathcal{T}_n(\infty) \rangle, \quad (3.11)$$

which implies trivial monodromy for the solutions of (3.3) (note that this expression does not depend on whether  $n$  is even or odd). The two-point function in (3.11) is proportional to  $(n-1)$  and thus vanishes in the limit  $n \rightarrow 1$ . Entanglement negativity is non-perturbatively small for  $x = 0$  as explained in [12].

For  $x = 1$  on the other hand,

$$\lim_{x \rightarrow 1} \langle \mathcal{T}_n(0) \overline{\mathcal{T}}_n(x) \overline{\mathcal{T}}_n(1) \mathcal{T}_n(\infty) \rangle = \langle \mathcal{T}_n(0) \overline{\mathcal{T}}_n^2(1) \mathcal{T}_n(\infty) \rangle, \quad (3.12)$$

The conformal dimension of  $\overline{\mathcal{T}}_n^2$  was found in [11-12] to be

$$\widehat{h}_n \equiv h_{\mathcal{T}_n^2} = \begin{cases} \frac{c}{24} \left( n - \frac{1}{n} \right), & n : \text{odd} \\ \frac{c}{12} \left( \frac{n}{2} - \frac{2}{n} \right), & n : \text{even} \end{cases} \quad (3.13)$$

To compute the negativity we are instructed to consider (3.12) for  $n$  even, analytically continue and take the limit  $n \rightarrow 1$ . In this limit the conformal dimension of the  $\mathcal{T}_n^2$  operator is

$$\widehat{h} \equiv \lim_{n_{\text{even}} \rightarrow 1} \widehat{h}_n = -\frac{c}{8} \quad (3.14)$$

For the three point function, one can explicitly determine all accessory parameters in (3.5). Solving (3.5) for the three point function in (3.12) with even  $n$  we obtain

$$T(z) = \frac{6h}{c} \frac{z^2 + (a-2)z + 1}{z^2(z-1)^2}, \quad a \equiv \frac{\widehat{h}}{h} = 2 \frac{\frac{n}{2} - \frac{2}{n}}{n - \frac{1}{n}}, \quad (3.15)$$



and  $h$  is defined in (2.9). The differential equation (3.3) can be solved analytically in the neighborhood of  $z \sim 1$ . The solutions read

$$\psi_{\pm}(z) \simeq (z-1)^{\frac{1}{2}} \left( 1 \pm \sqrt{1 - 24 \frac{\widehat{h}}{c}} \right), \quad (3.16)$$

and the trace of the monodromy matrix is  $\text{tr} M_{(x=1)} = -2 \cos \left[ \pi \sqrt{1 + \frac{4}{n} - n} \right]$ . When  $n \rightarrow 1$  the trace of the monodromy matrix reduces to

$$\text{tr} M_{(x,1)}|_{n \rightarrow 1} = -2 \quad (3.17)$$

We could also deduce this monodromy by setting  $n = 1$  in (3.13) evaluated for even  $n$ . This corresponds to  $\Lambda_p^2 = 4$  according to (3.7) which gives rise to (3.17).

In summary, to find the negativity in the vicinity of  $x = 1$  we need to solve eq. (3.3) with

$$T(z) = -\frac{c_2 x(1-x)}{z(z-x)(1-z)} \quad (3.18)$$

and impose the monodromy condition (3.17) which corresponds to an intermediate operator of conformal dimension  $h_p = \widehat{h}$  given in (3.14). We also need to ensure

$$c_2 \simeq -\frac{3}{4} \frac{1}{1-x}, \quad x \approx 1 \quad (3.19)$$

to recover (3.15) when  $x = 1$ . Entanglement negativity is then obtained from (3.1),

$$\frac{\partial \mathcal{E}}{\partial x} = \frac{c}{3} c_2(x) \quad (3.20)$$

It is relatively easy to recover (2.12) when  $x \rightarrow 1$ . In this limit the equation becomes

$$\psi''(z) + \frac{c_2(1-x)}{(z-1)^2} \psi(z) = 0 \quad (3.21)$$

and the choice of the accessory parameter (3.19) gives rise to the solutions  $\psi_1(z) \simeq (z-1)^{-\frac{1}{2}}$ ,  $\psi_2(z) \simeq (z-1)^{\frac{3}{2}}$  with the monodromy (3.17). Unfortunately we could not solve the general problem analytically, but we could do numerical integration. Below we describe the operational procedure. We start by rewriting the differential equation in the form

$$h'(z) + h^2(z) + T(z) = 0 \quad (3.22)$$

where  $\psi(z) = \exp(\int_{z_0}^z dz' h(z'))$ . To compute the monodromy, we integrate over a circle  $z = 1 + r_0 e^{2\pi i t}$ ,  $t \in (0, 1)$ . We make sure the contour encircles both 1 and  $x$ ,  $r_0 > (1 - x)$ . Eq. (3.22) then reads

$$\frac{\dot{h}}{2\pi i r_0 e^{2\pi i t}} + h^2 + T = 0 \quad (3.23)$$

where the dot denotes differentiation wrt  $t$ . We consider two independent solutions specified by the value of  $h(t = 0)$ :

$$h_1(t = 0) = 0, \quad h_2(t = 0) = 1 \quad (3.24)$$

This is translated to the following initial conditions at  $z_0 = 1 + r_0$ :

$$\psi_1(z_0) = 1, \quad \psi'_1(z_0) = h_1(t = 0) = 0 \quad (3.25)$$

and

$$\psi_2(z_0) = 1, \quad \psi'_2(z_0) = h_2(0) = 1 \quad (3.26)$$

On the other hand, we can integrate eq. (3.23) numerically to obtain the values of  $\psi_{1,2}$  at  $t = 1$ . Simple algebra then leads to the following expression for the monodromy trace:

$$\text{tr} M_{(x,1)} = (1 - a_1^{(1)})e^{a_2^{(1)}} + a_1^{(2)}e^{a_2^{(2)}} \quad (3.27)$$

where

$$a_1^{(i)} = h_i(t = 1), \quad a_2^{(i)} = 2\pi i r_0 \int_0^1 dt e^{2\pi i t} h_i(t), \quad i = 1, 2 \quad (3.28)$$

We perform numerical integration. Independence of  $r_0$  and limiting behavior (3.19) serve as simple checks. Another simple check involves comparing the solution of the monodromy problem with the expansion of the conformal block in terms of the cross-ratio [33]:

$$F(h_p, y) = y^{h_p} \left( 1 + \frac{1}{2} h_p y + \frac{h_p(h_p + 1)^2}{4(2h_p + 1)} y^2 + \frac{h_p^2(1 - h_p)^2}{2(2h_p + 1)(c(2h_p + 1) + 2h_p(8h_p - 5))} y^2 + \dots \right) \quad (3.29)$$

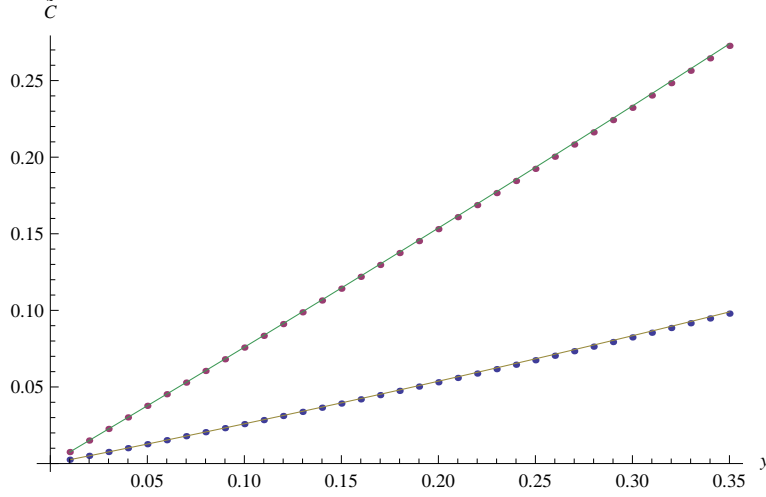
Here  $y = 1 - x$  and  $h_p$  is the dimension of the intermediate operator, what is called  $\Delta'$  in [33]. We also set  $\Delta$  of [33] to zero. It is now clear that for a generic  $h_p \simeq ac$  the first three terms in the bracket correspond to exponentiation

$$\exp\left(c(a \log y + \frac{a}{2}y + by^2 \dots)\right) \simeq y^\Delta \left(1 + \frac{ca}{2}y + \frac{c^2 a^2}{8}y^2 + cby^2 + \dots\right) \quad (3.30)$$

Comparing this with (3.29) implies

$$b = a \left( \frac{3}{16} + \frac{a}{8(1 + 8a)} \right) \quad (3.31)$$

which also agrees with [34-35]. We verified that our numerics reproduces the first three terms in the logarithm of the conformal block (3.2) for arbitrary generic values of  $a$ .



**Fig. 1:**  $\tilde{C} = 1 - C_2(y)/C_2(0)$  as a function of  $y$ . There are two solutions (dots) approximated by eq. (3.32).

Having checked our numerics we proceed to solve for the case of interest,  $h_p = -\frac{c}{8}$ . Contrary to the generic cases discussed above, we find *two* solutions (Fig.1). They are approximated by

$$C_2^\pm(y) = y(1-y)c_2(y) = -\frac{3}{4} \left( 1 - \left( \frac{1}{2} \pm \frac{1}{4} \right) y + \dots \right) \quad (3.32)$$

At large central charge we are instructed to choose the dominant solution, which corresponds to  $C_2^-$ . We then recover negativity by integrating (3.20). For sufficiently small  $x$  there is a phase transition to the other branch, where negativity is simply zero. (For small  $x$  the computation of the four-point function is identical to the one performed for entanglement entropy, and there is a factor of  $(n-1)$  which vanishes in the  $n \rightarrow 1$  limit.) We have checked that the two solutions never cross each other within the accuracy of our numerics.

#### 4. Discussion

We computed a conformal block which at large central charge gives rise to the value of the negativity for two disjoint intervals. We performed a numerical computation because we could not determine the accessory parameter analytically by solving the differential equation (3.3). It would be interesting to compare our result to the analytic expansion (3.29). However, precisely for  $h_p = ac = \hat{h} = -c/8$  (which corresponds to  $n = 1$  in (3.13), and which is the leading operator that appears in the OPE expansion of  $\mathcal{T}$  and  $\overline{\mathcal{T}}$ ) we encounter a complication. In this case the last term in (3.29) gets enhanced by a factor

of  $c$  and becomes  $\mathcal{O}(c^2 y^2)$ . This is related to the fact that our operator has a level two descendant which is null at leading order in central charge. Hence, the large  $c$  limit and the small  $x$  limits do not have to commute in this case.

One may wonder whether an analytic method for solving the Heun equation (3.3) order by order in  $y$  could be applied here (no issue with the order of limits in this case). A priori, there is no reason to doubt that this can be done. However, as we show in the appendix one finds the same difficulties as with the standard conformal block expansion. Namely, a divergent quadratic term. We believe that this hints to some non-analytic behavior of the conformal block. Indeed as explained above, the complication is traced to the fact that the intermediate operator has a null descendant at level two at leading order in the large  $c$  expansion. (This is true for all operators which correspond to  $tr M = \pm 2$  with the exception of the identity). In fact, there is an infinite tower of null states of even levels and it is natural to expect similar divergences to appear at higher orders in the expansion in powers of  $y^2$ ; the simplest non-analytic behavior resulting from the resummation could be  $\pm\sqrt{y^2} = \pm y$ , which would explain the origin of the  $\pm y/4$  term in (3.32). (From eq. (3.30) the coefficient of the linear term is  $\frac{3}{8}$ , in agreement with [34] and [35]. )

A related question is the validity of the derivation of (3.3) in our case. This derivation has been recently reviewed in the Appendix D of [31] and we do not find any fundamental differences when a null descendant is present in the intermediate channel at the leading order in the central charge expansion. One may wonder whether exponentiation of the conformal block (3.2) is still valid. It would be nice to address this question.

The method used in this paper does not allow the computation of the constant term in the value of negativity. In the case of mutual information [20] the holographic result could be used to predict a phase transition between the nonvanishing branch for  $x \geq 1/2$  and zero for  $x \leq 1/2$ . For negativity we also expect a phase transition, but cannot fix the value of  $x$  where it would happen. In any case, it would be interesting to have a holographic prescription for computing negativity<sup>1</sup>.

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<sup>1</sup> The behavior of the OPE coefficients at large central charge plays an important role in reproducing the holographic entanglement result. Their exponential dependence on the conformal dimension of the internal operator  $h_p$  may lead to the difference between the values of entanglement negativity in holographic theories and our result, which assumes that a certain conformal block gives a dominant contribution. We thank T. Faulkner and T. Hartman for discussions on this issue.

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## **Appendix A. Investigating the asymptotic expansion of the conformal blocks at large $c$ .**

There are two ways one may compute the conformal block at large  $c$ . One involves first expanding in  $x$  and then taking the large central limit, making use of (3.29). Another way is to first take the large  $c$  limit and then expand in  $x$ . The starting point in this case is the Heun equation (3.3). The latter approach was recently taken by the authors of [34] and [35] who attacked the problem in two distinct ways. As it turns out, whether one first takes the infinite central charge limit and then expand in the cross-ratio or vice versa is immaterial. The two limits appear to be commuting with each other and both methods yield the same result, *i.e.*, the infinite  $c$  limit of (3.29).

As explained in the Discussion section, we cannot directly apply (3.29) to our case. The intermediate operator we are interested in corresponds to a null state at large  $c$  and the quadratic correction in the cross-ratio  $x$  diverges. One might expect that approaching the problem using the methods of [34] and [35] adapted to our special case, would be possible to obtain a finite result. Here we see that this is not the case.

The authors of [34] showed that there is a one-to-one correspondence between conformal blocks at large central charge and the Painleve VI equation. To be precise they showed that the monodromy problem of the Heun equation can be mapped to the connection problem of the Painleve VI. Unfortunately, this approach cannot be successfully applied here, because the Painleve solution for our case is given by a Taylor series expansion. This fact

does not allow us to solve the connection problem order by order in  $x$ , as done in [34] for intermediate operators of generic dimension  $h_p$ .

We will thus use the method outlined in [35]. Let us review here the basic ingredients of the approach. Suppose that we wish to determine  $c_2$  such that the monodromy of the solutions of eq (3.3) on a path encircling both  $(0, x)$  is given by  $\text{tr}M = -2 \cos [\Lambda_p \pi]$ . We can expand  $T(z)$  for small  $x$  as<sup>2</sup>

$$T(z) = T_0 + T_1 x + T_2 x^2 + \mathcal{O}(x^3) \quad (\text{A.1})$$

where

$$\begin{aligned} T_0 &\equiv \frac{C_2(0)}{z^2(1-z)} \\ T_1 &\equiv \frac{C_2(0) + zC_2'(0)}{z^3(1-z)} \\ T_2 &\equiv \frac{2C_2(0) + 2zC_2'(0) + z^2C_2''(0)}{2z^4(1-z)}. \end{aligned} \quad (\text{A.2})$$

Since we are interested in the limit  $n \rightarrow 1$  we set  $h_i = 0$  and for convenience defined  $C_2(x) \equiv c_2(x)x(x-1)$  such that  $C_2(0) = \frac{6}{c}h_p = \frac{1}{4}(1 - \Lambda_p^2)$ . We now consider the differential equation (3.3) to leading order in  $x$ , namely,

$$\psi''(z) + \frac{C_2(0)}{z^2(1-z)}\psi(z) = 0 \quad (\text{A.3})$$

which for generic values of  $C_2(0)$  is solvable by means of simple hypergeometric functions. The key observation of [35] was that one can compute the monodromy matrix of the solutions of eq. (A.3) along a special path, depicted in Fig.2, for which  $T(z)$  is not singular. With this choice of contour, the monodromy matrix is easily determined by considering the behavior of the solutions at infinity. To leading order in  $x$  one finds that

$$\text{tr}M^{(0)} = -2 \cos [\Lambda_p \pi] \quad (\text{A.4})$$

which implies that all the corrections in  $\text{tr}M^{(0)}$  should vanish. These corrections can in principle be computed by solving eq.(A.3) order by order in  $x$ . Requiring that the  $\text{tr}(\delta M)$  vanishes at each order then determines  $C_2(x)$  as a series expansion in  $x$ .

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<sup>2</sup> Here for simplicity we discuss the case of small  $x$ . The case of small  $y = 1 - x$  can be treated in an identical manner.

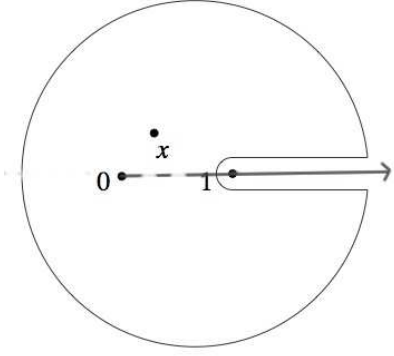


Fig.2. The contour.

This nice technique was developed in [35] where  $C_2(x)$  was determined up to quadratic order in  $x$ . The expressions obtained there are in complete agreement with [34]. As remarked earlier however, one cannot directly apply the formulas in [34-35] when  $\Lambda_p = 2$ . One reason this case is special can be traced to (A.3).

To be specific, when  $C_2(0) = -\frac{3}{4}$  the differential equation

$$\psi''(z) - \frac{3}{4z^2(1-z)}\psi(z) = 0, \quad (\text{A.5})$$

can be placed in the standard hypergeometric form

$$z(1-z)u''(z) + [c - (a+b+1)z]u'(z) - abu(z) = 0 \quad (\text{A.6})$$

with  $u(z) = z^{\frac{c}{2}}(1-z)^{1-c+a+b/2}\psi(z)$  and  $(a, b, c) = (\frac{3}{2}, \frac{1}{2}, 3)$ . However, when  $c$  is an integer, only one of the two independent solutions around  $z = 1$  is given by a standard hypergeometric function. For the other solution a much more complicated series representation exists. Here for simplicity we will use the following set of independent solutions

$$\begin{aligned} \psi_1(z) &= \frac{3}{4}z^{\frac{3}{2}}(1-z) {}_2F_1\left(\frac{3}{2}, \frac{5}{2}, 2, 1-z\right) \\ \psi_2(z) &= \frac{3i\pi^2}{16}z^{\frac{3}{2}} {}_2F_1\left(\frac{3}{2}, \frac{1}{2}, 3, z\right) \end{aligned} \quad (\text{A.7})$$

where the second solution is the standard solution around  $z = 0$ . Their wronskian is

$$w(\psi_1, \psi_2) \equiv \psi_1\psi_2' - \psi_1'\psi_2 = -\frac{3i\pi}{4}. \quad (\text{A.8})$$

It is sometimes convenient to express the solutions in terms of the complete elliptic integrals of the first and second kind, *i.e.*,

$$\begin{aligned} E[z] &= \int_0^{\frac{\pi}{2}} \sqrt{1 - z \sin^2 \phi} \\ K[z] &= \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - z \sin^2 \phi}} \end{aligned} \quad (\text{A.9})$$

in the following way

$$\begin{aligned}\psi_1(z) &= \frac{1}{\pi} z^{-\frac{1}{2}} ((z-2)E[1-z] + zK[1-z]) \\ \psi_2(z) &= -i\pi z^{-\frac{1}{2}} ((z-2)E[z] - 2(z-1)K[z])\end{aligned}\quad (\text{A.10})$$

The analytic continuation of  $\psi_1(z)$  for  $z > 1$  above and below the axis can be simply written as

$$\psi_1^\pm(z) = \frac{3}{4} z^{\frac{3}{2}} e^{\mp i\pi} (z-1) {}_2F_1\left(\frac{3}{2}, \frac{5}{2}, 2, 1-z\right) \equiv -g_1(z) \quad (\text{A.11})$$

For the other solution we write

$$\psi_2^\pm(z) = -i\pi z^{-\frac{1}{2}} ((z-2)E[-e^{\mp i\pi} z] - 2(z-1)K[-e^{\mp i\pi} z]) \quad (\text{A.12})$$

To compute the monodromy matrix to leading order in  $x$  we need the asymptotic behavior of the solutions at infinity. To find the behavior for large  $|z| \gg 1$  it is convenient to use the known asymptotic expansions of the elliptic integrals and substitute  $z$  by  $z \rightarrow -ze^{\pm i\pi}$ . The result is

$$\begin{aligned}\psi_1^\pm(z) &= -\frac{z}{\pi} - \frac{3}{4\pi} \ln z + \frac{3}{4\pi} (3 - 4 \ln 2) + \mathcal{O}(z^{-1} \ln z) \\ \psi_2^\pm(z) &= \mp \pi z + \mp \frac{3\pi}{4} \ln z + \pm \frac{3\pi}{4} (3 - 4 \ln 2) + \frac{3i\pi^2}{4} + \mathcal{O}(z^{-1} \ln z)\end{aligned}\quad (\text{A.13})$$

Denoting by  $\Upsilon^\pm \equiv \begin{pmatrix} \psi_1^\pm \\ \psi_2^\pm \end{pmatrix}$  the matrix of the two independent solutions, we can express the asymptotic behavior of the solution continued slightly *above* the real axis as

$$\Upsilon_\infty^+ \simeq B^+ \Upsilon_0 = \begin{pmatrix} -\frac{1}{\pi} & 0 \\ -\pi & \frac{3}{4}i\pi^2 \end{pmatrix} \begin{pmatrix} z + \frac{3}{4} \ln z - \frac{3}{4}(3 - 4 \ln 2) \\ 1 \end{pmatrix} \quad (\text{A.14})$$

and that of the solution continued *below* the real axis as

$$\Upsilon_\infty^- \simeq B^- \Upsilon_0 = \begin{pmatrix} -\frac{1}{\pi} & 0 \\ \pi & \frac{3}{4}i\pi^2 \end{pmatrix} \begin{pmatrix} z + \frac{3}{4} \ln z - \frac{3}{4}(3 - 4 \ln 2) \\ 1 \end{pmatrix} \quad (\text{A.15})$$

We can now start from the solution defined at  $z = \infty - i\epsilon$ , eq.(A.15) and take a full turn around infinity to go back to  $z = \infty - i\epsilon$ , but having now encircled both the origin and  $x$  as can be seen in figure 1. The turn around infinity yields

$$\tilde{\Upsilon}_\infty^- \simeq \tilde{B}^- \Upsilon_0 = \begin{pmatrix} -\frac{1}{\pi} & \frac{3}{2}i \\ \pi & -\frac{3}{4}i\pi^2 \end{pmatrix} \begin{pmatrix} z + \frac{3}{4} \ln z - \frac{3}{4}(3 - 4 \ln 2) \ln z \\ 1 \end{pmatrix} \quad (\text{A.16})$$



where the change in the constant terms is due to the change in the logarithm. We now observe that

$$\Upsilon^+ = \left( B^+ (\tilde{B}^-)^{-1} \right) \tilde{\Upsilon}^- \quad (\text{A.17})$$

which allows us to compute the monodromy matrix to leading order in  $x$ ,

$$M^{(0)} = \left( B^+ (\tilde{B}^-)^{-1} \right) = \begin{pmatrix} -1 & -\frac{2}{\pi^2} \\ 0 & -1 \end{pmatrix} \quad (\text{A.18})$$

and confirm that  $\text{tr} M^{(0)} = -2$ .

Consider now the monodromy matrix  $M$  of the solutions of eq. (3.3) and its expansion in powers of  $x$

$$M = M^{(0)} + x\delta M^{(1)} + x^2\delta M^{(2)} + \mathcal{O}(x^3) \quad (\text{A.19})$$

Since the desired trace  $\text{tr} M = \text{tr} M^{(0)} = -2$  is achieved at leading order, it must be that  $\text{tr} \delta M^{(1)} = \text{tr} \delta M^{(2)} = \dots = 0$ . The correction to the monodromy matrix at each order can be evaluated from the asymptotic behavior of the “corrected” solutions at infinity, *i.e.*,

$$\begin{aligned} \Upsilon_\infty^+ + x\delta\Upsilon_\infty^{(1),+} + x^2\delta\Upsilon_\infty^{(2),+} + \dots = \\ = \left( M^{(0)} + x\delta M^{(1)} + x^2\delta M^{(2)} + \dots \right) \left( \tilde{\Upsilon}_\infty^- + x\delta\tilde{\Upsilon}_\infty^{(1),-} + x^2\delta\tilde{\Upsilon}_\infty^{(2),-} + \dots \right) \end{aligned} \quad (\text{A.20})$$

Once  $\delta M^{(i)}$  is expressed in terms of  $C^i(0)$ , solving  $\delta M^{(i)} = 0$  computes  $C^i(0)$ .

In what follows we focus on the first and second order correction to the monodromy matrix. We start by considering equation (3.3) to first order in  $x$ ,

$$\psi''(z) + T_0(z)\psi(z) + x T_1(z)\psi(z) = 0, \quad (\text{A.21})$$

where  $T_0, T_1$  are given in (A.2). The solutions of (A.21) are of the form  $\psi_i(z) + x\delta\psi_i(z)$  where  $\psi_i$  are the solutions of the zeroth order in  $x$  differential equation  $\psi'' + T_0\psi = 0$  and  $\delta\psi_i$  are the solutions of the inhomogeneous differential equation

$$\delta\psi'' + T_0\delta\psi = -T_1\psi \quad (\text{A.22})$$

which can be expressed as follows

$$\delta\psi_i(z) = \frac{\psi_1(z)}{w_{12}} \int_1^z dz' \psi_2(z') T_1(z') \psi_i(z') - \frac{\psi_2(z)}{w_{12}} \int_1^z dz' \psi_1(z') T_1(z') \psi_i(z'). \quad (\text{A.23})$$

Focusing on the asymptotic behavior of the solutions of (A.21) at infinity, we find that

$$\delta\Upsilon_\infty^\pm = \Lambda_1^\pm \Upsilon_\infty^\pm, \quad \delta\tilde{\Upsilon}_\infty^- = \Lambda_1^- \tilde{\Upsilon}_\infty^-, \quad (\text{A.24})$$

where  $\Lambda_1^\pm$  is the following matrix

$$\Lambda_1^\pm = \begin{pmatrix} \Delta_{12}^\pm & -\Delta_{11}^\pm \\ \Delta_{22}^\pm & -\Delta_{12}^\pm \end{pmatrix}, \quad \Delta_{ij} = \frac{1}{w_{12}} \int_1^\infty dz \psi_i(z) T_1(z) \psi_j(z). \quad (\text{A.25})$$

Substituting (A.24) into (A.20) leads to

$$\delta M^{(1)} = \left( \delta Y_\infty^+ - M^{(0)} \delta \tilde{Y}_\infty^- \right) = \Lambda_1^+ M^{(0)} - M^{(0)} \Lambda_1^-, \quad (\text{A.26})$$

while using (A.18) and taking the trace yields

$$\text{tr} \delta M^{(1)} = -\frac{2}{\pi^2} (\Delta_{22}^+ - \Delta_{22}^-) = 0 \quad (\text{A.27})$$

With the help of eq. (A.51) we can express (A.27) as follows

$$U_{22}^+(f_0) - U_{22}^-(f_0)|_1^\infty = 0 \quad (\text{A.28})$$

where

$$f_0(v) = \frac{4}{3} C_2'(0)(z-1) + \left( 1 - \frac{8}{3} C_2'(0) \right) \frac{z-1}{z}. \quad (\text{A.29})$$

and  $U_{ij}$  is defined in (A.52). Using the asymptotic expressions for the solutions  $\psi^\pm$  at  $z=1$  and  $z=\infty$  given in Appendix A.3, the divergent terms cancel each other and we obtain

$$C_2'(0) = \frac{3}{8}, \quad (\text{A.30})$$

which matches the standard result for the conformal block at large  $c$  for  $\Lambda_p = -\frac{3}{4}$ .

For the quadratic correction to the accessory parameter, we need to consider second order corrections to the differential eq. (A.5), namely,

$$\psi''(z) + T_0(z)\psi(z) + xT_1(z)\psi(z) + x^2T_2(z)\psi(z) = 0 \quad (\text{A.31})$$

with  $T_0, T_1, T_2$  given in (A.2). The solutions of (A.31) are

$$\psi_i^{(2)}(z) = \psi_i(z) + x\delta\psi_i^{(1)}(z) + x^2\delta\psi_i^{(2)}(z) \quad (\text{A.32})$$

with  $\delta\psi_i^{(1)}$  as in (A.23) and  $\delta\psi_i^{(2)}$  equal to

$$\delta\psi_i^{(2)} = \frac{\psi_1}{w_{12}} \left( \int_1^z \psi_2 T_2 \psi_i + \int_1^z \psi_2 T_1 \delta\psi_i^{(1)} \right) - \frac{\psi_2}{w_{12}} \left( \int_1^z \psi_1 T_2 \psi_i + \int_1^z \psi_1 T_1 \delta\psi_i^{(1)} \right). \quad (\text{A.33})$$

To find the asymptotic behavior of the solutions at infinity note that

$$\delta\Upsilon_\infty^{(2)\pm} = \Lambda_2^\pm \Upsilon_\infty^\pm, \quad \delta\tilde{\Upsilon}_\infty^{(2)-} = \Lambda_2^- \tilde{\Upsilon}_\infty^- \quad (\text{A.34})$$

where

$$\Lambda_2^\pm = Z^\pm + \Theta^\pm, \quad Z^\pm = \begin{pmatrix} Z_{12}^\pm & -Z_{11}^\pm \\ Z_{22}^\pm & -Z_{12}^\pm \end{pmatrix}, \quad \Theta^\pm = \begin{pmatrix} \Theta_{12}^\pm & -\Theta_{11}^\pm \\ \Theta_{22}^\pm & -\Theta_{21}^\pm \end{pmatrix} \quad (\text{A.35})$$

and

$$Z_{ij}^\pm = \frac{1}{w_{12}} \int_1^\infty \psi_i^\pm T_2 \psi_j^\pm, \quad \Theta_{ij}^\pm = \frac{1}{w_{12}} \int \delta\psi_i^{(1)\pm} T_1 \psi_j^\pm \quad (\text{A.36})$$

Using eqs. (A.34), (A.36) together with the relations  $\Delta_{11}^+ = \Delta_{11}^-$  (which is due to (A.11)) and  $\Delta_{22}^+ = \Delta_{22}^-$  (which ensures that  $\delta M^{(1)} = 0$ ), results in

$$\begin{aligned} \text{tr}\delta M^{(2)} = & \frac{2}{\pi^2} [(\Delta_{12}^+ - \Delta_{12}^-)(\Delta_{22}^- + \pi^2 \Delta_{12}^-) + \\ & + \frac{1}{2} ((\Theta_{12}^- - \Theta_{21}^-) - (\Theta_{12}^+ - \Theta_{21}^+)) + ((\Theta_{22}^- + Z_{22}^-) - (\Theta_{22}^+ + Z_{22}^+))] \end{aligned} \quad (\text{A.37})$$

With the help of Appendix A we can express the different terms appearing in (A.37) as follows

$$\begin{aligned} w_{12}(\Delta_{12}^+ - \Delta_{12}^-)(\Delta_{22}^- + \pi^2 \Delta_{12}^-) &= (U_{12}^+(f_0) - U_{12}^-(f_0))(U_{22}^-(f_0) + \pi^2 U_{12}^-(f_0)) \Big|_1^\infty \\ w_{12}(\Theta_{12}^\pm - \Theta_{21}^\pm) &= S_{12}^\pm(f_0, g_0) - S_{21}^\pm(f_0, g_0) \Big|_1^\infty \\ w_{12}(\Theta_{22}^\pm + Z_{22}^\pm) &= S_{22}^\pm(f_0, g_0) + \frac{9}{16} \left( S_{22}^\pm(f_4, g_4) - S_{22}^\pm(\hat{f}_4, \hat{g}_4) - \mathcal{S}_{22}^\pm \right) - C_2''(0) \frac{1}{2} U_{22}^\pm(f_2) \Big|_1^\infty \end{aligned} \quad (\text{A.38})$$

where  $U_{ij}(f), S_{ij}(f, g), \mathcal{S}_{ij}$  are defined in (A.52), (A.61) and (A.73) whereas

$$\begin{aligned} f_0 &= \frac{1}{2}(z-1), & g_0 &= \frac{1}{4}(z-1) - \frac{1}{2} \frac{z-1}{z}, & f_2 &= -\frac{4}{3}z + 4 - \frac{8}{3z} \\ f_4 &= \frac{4}{3}(1 - \frac{1}{z}), & g_4 &= \frac{16}{9} - \frac{8}{9} \frac{1}{z^3} + \frac{4}{9} \frac{1}{z^2} - \frac{4}{3} \frac{1}{z}, \\ \hat{f}_4 &= \frac{z}{3} + \frac{1}{3} - \frac{2}{3z}, & \hat{g}_4 &= \frac{16}{9} - \frac{13}{9z} - \frac{1}{9z^2} - \frac{2}{9z^3} \end{aligned} \quad (\text{A.39})$$

Finally, using the asymptotic expressions for the solutions  $\psi_i^\pm$  as well as for  $p_i^\pm, q_i^\pm$  at  $z = 1$  and  $z = \infty$  given in Appendix A.3 we find that all divergent terms cancel out except for the ones coming from the first order terms, *i.e.*,  $(\Delta_{12}^+ - \Delta_{12}^-)(\Delta_{22}^- + \pi^2 \Delta_{12}^-)$  (see (A.82)). Hence the divergence at quadratic order persists, hinting at the non-analyticity of the accessory parameter  $C_2(x)$ .

### A.1. Analytic computation of integrals with hypergeometric functions.

The following integrals are necessary for computing the first and second order corrections to the accessory parameter  $C_2(x)$

$$I_{m,ij}^{\pm}(z) = \int_1^z dz \frac{\psi_i^{\pm}(z)\psi_j^{\pm}(z)}{z^m(z-1)}, \quad m = 2, 3, 4, \quad (\text{A.40})$$

and

$$\delta I_{ij}^{\pm}(z) = \int_1^z \delta \psi_i^{(1),\pm} T_1 \psi_j^{\pm}, \quad T_1 = -\frac{3}{8} \frac{1}{z^2(z-1)} + \frac{3}{4} \frac{1}{z^3(z-1)}. \quad (\text{A.41})$$

A method for computing such integrals was developed in [35]. Here we will use and expand this method as is necessary to treat the special case  $\Lambda_p = 2$ . We should mention here that such integrals are in general divergent. We will see that in most divergences cancel when evaluating the corrections to the accessory parameter.

Consider a coordinate transformation from the  $z$ -variable to some new variable  $v$  which also depends on some arbitrary parameter  $a$  such that

$$z \equiv z(v, a) = v + f(v)a + g(v)a^2 + \mathcal{O}(a^3). \quad (\text{A.42})$$

The zeroth order differential equation  $\psi''(z) + T_0(z)\psi(z) = 0$  under this change of variable becomes

$$\ddot{t}(v, a) + R(v, a)t(v, a) = 0, \quad (\text{A.43})$$

where the dots denote differentiation with respect to  $v$  and

$$\begin{aligned} R(v, a) &= \left( \frac{\partial z}{\partial v} \right)^2 T_0(z(v, a)) + \frac{1}{2} \{z, v\}, \\ t_i(v, a) &= \left( \frac{\partial z(v, a)}{\partial v} \right)^{-\frac{1}{2}} \psi_i(z(v, a)). \end{aligned} \quad (\text{A.44})$$

Here  $\{z, v\}$  is the Schwartzian of the transformation defined as follows

$$\{z, v\} \equiv \frac{\ddot{z}}{\dot{z}} - \frac{3}{2} \left( \frac{\ddot{z}}{\dot{z}} \right)^2. \quad (\text{A.45})$$

From the definition and properties of the transformation (A.42) and (A.44) it follows that

$$R(v, a) = T_0(v) + aR_1(v) + a^2R_2(v) + \mathcal{O}(v^3) \quad (\text{A.46})$$

with

$$\begin{aligned}
R_1(v) &= \frac{(6-9v)f(v) + 2v(v-1)(3f'(v) + v^2(v-1)f'''(v))}{4v^3(v-1)^2} \\
R_2(v) &= \frac{(6-9v)g(v) + 2v(v-1)(3g'(v) + v^2(v-1)g'''(v))}{4v^3(v-1)^2} + \\
&+ \frac{3(3-8v+6v^2)f(v)^2 - 6v(2-5v+3v^2)f'(v)f(v)}{4v^4(v-1)^3} + \\
&+ \frac{v^2(v-1)^2(3f'(v)^2 - 3v^2(v-1)f''(v)^2 - 2(v-1)v^2f'(v)f'''(v))}{4v^4(v-1)^3}
\end{aligned} \tag{A.47}$$

Note also that  $\lim_{a \rightarrow 0} t(v, a) = \psi(v)$ .

Let us now differentiate (A.43) with respect to  $a$  and take the limit  $a \rightarrow 0$  to obtain

$$\ddot{\phi}_i + T_0 \dot{\phi}_i + R_1 \psi_i = 0, \tag{A.48}$$

where we set

$$\phi_i(v) \equiv \left. \frac{\partial t_i(v, a)}{\partial a} \right|_{a=0}, \tag{A.49}$$

and used  $t_i(v, a=0) = \psi_i(v)$ . Multiplying (A.48) with  $\psi_j$  and integrating by parts leads to<sup>3</sup>

$$\int_1^z \psi_j R_1 \psi_i = \psi_j' \phi_i(z) - \psi_j \phi_i'(z) \Big|_1^z \tag{A.50}$$

Further using (A.42) and (A.44) to evaluate  $\phi_i$  yields

$$\int_1^z \psi_j R_1 \psi_i = U_{ij}(f) \Big|_1^z \tag{A.51}$$

where we defined

$$U_{ij}(f) = f(z) (\psi_i' \psi_j' - \psi_i \psi_j'') - \frac{1}{2} f'(z) (\psi_i' \psi_j + \psi_i \psi_j') + \frac{1}{2} f''(z) \psi_i \psi_j \tag{A.52}$$

To confirm that  $U_{ij}(f)$  is symmetric under the exchange  $\psi_i \rightarrow \psi_j$ , recall that  $\psi_i \psi_j'' = \psi_i'' \psi_j$  for two solutions  $\psi_{i,j}$  with constant wronskian (A.8).

Computing integrals of the type (A.40) is now a straightforward exercise; one simply needs to choose  $R_1(v)$ , or rather  $f(v)$ , appropriately and employ (A.51). Thus the original problem has been mapped to the problem of finding a particular solution to the inhomogeneous differential equation which determines  $f(v)$  for a specific choice of the function  $R_1(v)$ , *i.e.*, (A.46).

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<sup>3</sup> We assume that  $\left. \frac{\partial}{\partial a} \frac{\partial}{\partial v} \right|_{a=0} = \left. \frac{\partial}{\partial v} \frac{\partial}{\partial a} \right|_{a=0}$

Finding a suitable  $f(v)$  for the integrals in (A.40) with  $m = 2, 3$  is relatively easy. The choices  $f(v) = -\frac{4}{3}v + 4 - \frac{8}{3v}$  and  $f(v) = \frac{4}{3}(1 - \frac{1}{v})$  respectively yield

$$\begin{aligned} R_1 &= \frac{1}{v^2(v-1)}, & f(v) &= f_2(v) \equiv -\frac{4}{3}v + 4 - \frac{8}{3v} \\ R_1 &= \frac{1}{v^3(v-1)}, & f(v) &= f_3(v) \equiv \frac{4}{3}(1 - \frac{1}{v}). \end{aligned} \quad (\text{A.53})$$

It is also possible to directly compute the necessary linear combination of integrals appearing in  $\Delta_{ij}$  (defined in (A.25)) by choosing  $f(v) = f_0(v) \equiv \frac{4C'_2(0)}{3}(v-1) + \frac{3-8C'_2(0)}{3} \frac{v-1}{v}$  such that  $R_1(v) = T_1(v)$ .

The case  $m = 4$  in (A.40) is special because a simple solution for  $f(v)$  cannot be easily found. We will describe a slightly different technique for evaluating  $I_4$  in the following subsection. We conclude here by explaining how to compute integrals like the one in (A.41).

The starting point is again eq.(A.43) which we now differentiate twice with respect to  $a$  to obtain

$$\ddot{\chi}_i + T_0\chi_i + 2R_1\phi_i + 2R_2\psi_i = 0. \quad (\text{A.54})$$

In writing (A.54) we used the following definitions

$$\chi(v) \equiv \left. \frac{\partial^2 t(v, a)}{\partial a^2} \right|_{a=0}, \quad \phi(v) \equiv \left. \frac{\partial t(v, a)}{\partial a} \right|_{a=0}, \quad t(v, a)|_{a=0} = \psi(v). \quad (\text{A.55})$$

Let us note here that  $\phi$  satisfies a differential equation similar to the one  $\delta\psi$  satisfies. In fact, when  $R_1 = T_1$ , (A.48) coincides with (A.22) and as a result  $\phi_i$  can be identified with  $\delta\psi_i$ . It is now clear how (A.54) can help us compute (A.41). We first choose  $f(v)$  in such a way that  $R_1 = T_1$  and  $\phi = \delta\psi_i^{(1)}$ . As previously mentioned, this can be achieved by

$$f(v) = f_0(v) = \frac{4C'_2(0)}{3}(v-1) + \frac{3-8C'_2(0)}{3} \frac{v-1}{v}, \quad \Rightarrow_{C'_2(0)=\frac{3}{8}} \quad f_0(v) = \frac{1}{2}(v-1). \quad (\text{A.56})$$

We then multiply (A.54) with  $\psi_j$  and integrate by parts to obtain

$$\int_1^z \delta\psi_i^{(1)} T_1 \psi_j = \frac{1}{2} (\psi_j' \chi_i - \psi_j \chi_i') \Big|_1^z - \int_1^z \psi_i R_2 \psi_j. \quad (\text{A.57})$$

As long as we can evaluate  $\int_1^z \psi_i R_2 \psi_j$ , we can also evaluate (A.57). Note however that  $R_2$  can take any form we like since (A.47), as a linear differential equation for  $g(v)$ , admits a solution for any inhomogeneous term, at least in principle. Here we choose

$$g(v) = g_0(v) \equiv \frac{1}{4}(v-1) - \frac{1}{2} \frac{v-1}{v}, \quad (\text{A.58})$$

which results in

$$R_2 = \frac{9}{16} \frac{1}{v^4(v-1)} - \frac{3}{8} \frac{1}{v^3(v-1)}. \quad (\text{A.59})$$

With the choice (A.58) the integral on the right hand side of (A.57) is of the type of (A.40) with  $m = 3, 4$ , which we know how to evaluate. Finally, combining (A.59) together with (A.56) and (A.57) while taking into account (A.42) and (A.44) leads to

$$\int_1^z \delta\psi_i^{(1)} T_1 \psi_j = S_{ij}(f_0, g_0) - \frac{9}{16} I_{4(ij)} + \frac{3}{8} I_{3(ij)} \quad (\text{A.60})$$

where  $S_{ij}(f, g)$  is defined as

$$\begin{aligned} S_{ij}(f, g) \equiv & \frac{1}{2} \psi_j' \left[ \psi_i \left( \frac{3}{4} f'^2 - g' \right) + \psi_i' (2g - f f') + f^2 \psi_i'' \right] \Big|_1^z - \\ & - \frac{1}{2} \psi_j \left[ \psi_i \left( \frac{3}{2} f' f'' - g'' \right) + \psi_i' \left( g' - \frac{1}{4} f'^2 - f f'' \right) + \psi_i'' (2g + f f') + f^2 \psi_i''' \right] \Big|_1^z \end{aligned} \quad (\text{A.61})$$

## A.2. Computing the integral $I_4$ .

Consider the following integral

$$I_3(\phi, \psi) \equiv \int_1^z dv \frac{\phi_i(v) \psi_j(v)}{v^3(v-1)}, \quad (\text{A.62})$$

where  $\psi_i$  is a solution of (A.5) and  $\phi_i$  satisfies the inhomogeneous differential equation

$$\phi_i'' + T_0 \phi_i + V_1 \psi_i = 0. \quad (\text{A.63})$$

with  $V_1 = \frac{1}{z^3(z-1)}$ .

We explained how to evaluate integrals of this form at the beginning of Appendix A. We start from the original differential equation (A.5) and perform a change variable as in (A.42), choosing  $f(v)$  such that  $R_1$  in (A.46) coincides with  $V_1$ . We then pick  $g(v)$  such that the integral  $\int_1^z \psi_i \psi_j R_2$  is expressed in terms of the known integrals  $I_2, I_3$  and/or the sought for integral  $I_4$ . In practice we find that a convenient choice is

$$f(v) = \frac{4}{3} \left( 1 - \frac{1}{v} \right) \equiv f_4(v), \quad g(v) = \frac{16}{9} - \frac{8}{9} \frac{1}{v^3} + \frac{4}{9} \frac{1}{v^2} - \frac{4}{3} \frac{1}{v} \equiv g_4(v) \quad (\text{A.64})$$

and leads to

$$\int_1^z \frac{\phi_i \psi_j}{v^3(v-1)} = S_{ij}(f_4, g_4) - \frac{7}{3} I_4 \quad (\text{A.65})$$

where  $S_{ij}(f, g)$  is defined in (A.61).

Consider now an alternative, slightly more involved method to evaluate the same integral, (A.62). We start from (A.5) but instead of performing a change of variable, we induce a change in  $T_0$  parametrized by  $a$ , *i.e.*,

$$u'' + \frac{3}{4z^2(z-1)}(1 + as_0)u = 0, \quad (\text{A.66})$$

with  $s_0$  an arbitrary number. The two independent solutions of (A.66) can be expressed in terms of hypergeometric functions as follows

$$\begin{aligned} u_1(a, z) &= \frac{3}{4}(z-1)z^{\frac{1+\sqrt{4+3as_0}}{2}} {}_2F_1 \left[ \frac{1+\sqrt{4+3as_0}}{2}, \frac{3+\sqrt{4+3as_0}}{2}, 2, 1-z \right] \\ u_2(a, z) &= \frac{3}{16}i\pi^2 z^{\frac{1+\sqrt{4+3as_0}}{2}} {}_2F_1 \left[ \frac{-1+\sqrt{4+3as_0}}{2}, \frac{1+\sqrt{4+3as_0}}{2}, 1+\sqrt{4+3as_0}, z \right]. \end{aligned} \quad (\text{A.67})$$

The basis of the solutions is selected so that it reduces to (A.7) when  $s_0 = 0$ . Taylor expansion around the point  $a = 0$  yields  $u(a, z) = \psi(z) + ap(z) + a^2q(z) + \mathcal{O}(a^2)$ , where the functions  $p(z), q(z), \dots$  can be computed explicitly.

Next we change variables from  $z$  to  $v$  according to (A.42) with the same parameter  $a$  appearing in (A.66). We are interested in producing an additional relation between the integrals  $I_4$  and  $I_3(\phi, \psi)$ , we thus choose the pair  $(f, g)$  such that  $R_1$  in (A.46) is *identical* to  $V_1$  and  $R_2$  is proportional to  $\frac{1}{v^4(v-1)}$ . It is straightforward to verify that the following set of functions  $(\hat{f}_4, \hat{g}_4)$  satisfy the above requirements, *i.e.*,

$$\begin{aligned} f(v) &= s_0v + \frac{1}{3}(4 - 9s_0) + \frac{2}{3}(-2 + 3s_0)\frac{1}{v} \equiv \hat{f}_4(v), \\ g(v) &= \left( \frac{16}{9} + \frac{4}{3}s_0 - 4s_0^2 \right) - \left( \frac{4}{3} + \frac{4}{3}s_0 - 3s_0^2 \right) \frac{1}{v} + \left( \frac{4}{9} - \frac{8}{3}s_0 + 3s_0^2 \right) \frac{1}{v^2} - \\ &\quad - \left( \frac{8}{9} - \frac{8}{3}s_0 + 2s_0^2 \right) \frac{1}{v^3} \equiv \hat{g}_4(v). \end{aligned} \quad (\text{A.68})$$

Following the approach of Appendix A we differentiate (A.66) twice with respect to  $a$  and evaluate at  $a = 0$ , which yields

$$\ddot{\chi}_i + T_0\chi_i + 2R_1\phi_i + 2R_2\psi_i = 0, \quad (\text{A.69})$$

with

$$\chi_i \equiv \left. \frac{\partial^2 t_i}{\partial a^2} \right|_{a=0}, \quad \phi_i \equiv \left. \frac{\partial t_i}{\partial a} \right|_{a=0}, \quad R_1 = \frac{1}{v^3(v-1)}, \quad R_2 = \frac{7-9s_0^2}{3v^4(v-1)}. \quad (\text{A.70})$$



Once more,  $R_1$  being equal to  $V_1$  implies that  $\phi_i$  in (A.69) is a solution of (A.63). Multiplying with  $\psi_j$  and integrating leads to

$$\begin{aligned} \int_1^z \frac{\phi_i \psi_j}{v^3(v-1)} &= \frac{1}{2} \left( \dot{\psi}_j \chi_i - \psi_j \dot{\chi}_i \right) \Big|_1^z - \frac{1}{3} (7 - 9s_0^2) I_{4,ij} = . \\ &= \frac{1}{2} \dot{\psi}_j \left( 2q_i - p_i \dot{f}_2 + 2\dot{p}_i f_2 \right) - \frac{1}{2} \psi_j \left( 2\dot{q}_i + \dot{p} \dot{f}_2 - p_i \ddot{f}_2 + 2\ddot{p}_i f_2 \right) + \\ &+ S_{ij}(f_2, g_2) - \frac{1}{3} (7 - 9s_0^2) I_{4,ij} \end{aligned} \quad (\text{A.71})$$

$I_4$  is then readily computed using the linearly independent equations (A.71) and (A.65),

$$I_{4,(ij)} = \frac{S_{ij}(f_4, g_4) - S_{ij}(\widehat{f}_4, \widehat{g}_4) - \mathcal{S}_{ij}}{3s_0^2} \quad (\text{A.72})$$

where we defined

$$\mathcal{S}_{ij} \equiv \frac{1}{2} \dot{\psi}_j \left( 2q_i - p_i \dot{f}_2 + 2\dot{p}_i f_2 \right) - \frac{1}{2} \psi_j \left( 2\dot{q}_i + \dot{p} \dot{f}_2 - p_i \ddot{f}_2 + 2\ddot{p}_i f_2 \right) \quad (\text{A.73})$$

Thus far, we kept  $s_0$  arbitrary. In the actual computation, we chose  $s_0 = \frac{1}{3}$ .

We have finally shown how to compute all the integrals necessary for the evaluation of the quadratic corrections to the accessory parameter  $C_2$ . The rest is a matter of bookkeeping.

### A.3. List of asymptotic expressions.

#### 1 . Asymptotics of the solutions

$$\begin{aligned} \psi_1^\pm(z) &\simeq_{z \sim \infty} -\frac{z}{\pi} - \frac{3}{4\pi} \ln z + \frac{3}{4\pi} (3 - 4 \ln 2) + \frac{9}{32\pi} \frac{\ln z}{z} - \frac{3}{64\pi} (1 - 24 \ln 2) \frac{1}{z} + \dots \\ \psi_1^\pm(z) &\simeq_{z \sim 1} -\frac{3}{4}(z-1) + \frac{9}{32}(z-1)^2 + \dots \end{aligned} \quad (\text{A.74})$$

$$\begin{aligned} \psi_2^\pm(z) &\simeq_{z \sim \infty} \mp \pi z + \mp \frac{3\pi}{4} \ln z \pm \frac{3\pi}{4} (3 - 4 \ln 2) + \frac{3}{4} i\pi^2 \\ &\pm \frac{9\pi}{32} \frac{\ln z}{z} + \left( \mp \frac{3\pi}{64} (1 - 24 \ln 2) - \frac{9i\pi^2}{32} \right) \frac{1}{z} + \dots \\ \psi_2^\pm &\simeq_{z \sim 1} i\pi - \frac{3i\pi}{4} \ln[z-1](z-1) + \left( \frac{i\pi}{4} (-5 + 12 \ln 2) \mp \frac{3\pi^2}{4} \right) (z-1) + \dots \end{aligned} \quad (\text{A.75})$$

#### 2 . Asymptotics of $p_2^\pm(z)$ .

$$\begin{aligned} p_2^\pm &\simeq_{z \sim \infty} p_{21}^\pm z + p_{22}^\pm \ln z + p_{23}^\pm + \dots \\ p_2^\pm &\simeq_{z \sim 1} P_{21}^\pm + P_{22}^\pm (z-1) \ln[z-1] + P_{23}^\pm (z-1) + \dots \end{aligned} \quad (\text{A.76})$$

where

$$\begin{aligned}
p_{21}^{\pm} &= \pm \frac{\pi}{24} (5 - 12 \ln 2) - \frac{i\pi^2}{8} \\
p_{22}^{\pm} &= \pm \frac{3\pi}{32} (-1 - 8 \ln 2) - \frac{3i\pi^2}{32} \\
p_{23}^{\pm} &= \pm \frac{3\pi}{32} (-5 - 8 \ln 2(-1 + 2 \ln 2)) + \frac{3i\pi^2}{8}
\end{aligned} \tag{A.77}$$

and

$$\begin{aligned}
P_{21}^{\pm} &= \frac{i\pi}{24} (-5 + 12 \ln 2) \\
P_{22}^{\pm} &= -\frac{3i\pi}{32} (1 + 4 \ln 2) \\
P_{23}^{\pm} &= \pm \frac{3\pi^2}{32} (-1 - 4 \ln 2) - \frac{i\pi}{96} (-25 + 9\pi^2 + 24(1 - 6 \ln 2) \ln 2)
\end{aligned} \tag{A.78}$$

3 . Asymptotics of  $q_2^{\pm}(z)$ .

$$\begin{aligned}
q_2^{\pm} &\simeq_{z \sim \infty} q_{21}^{\pm} z + q_{22}^{\pm} \ln z + q_{23}^{\pm} + \dots \\
q_2^{\pm} &\simeq_{z \sim 1} Q_{21}^{\pm} + Q_{22}^{\pm} (z - 1) \ln [z - 1] + Q_{23}^{\pm} (z - 1) + \dots
\end{aligned} \tag{A.79}$$

where

$$\begin{aligned}
q_{21}^{\pm} &= \pm \frac{\pi}{1152} (-71 + 12\pi^2 + 156 \ln 2 - 144 \ln^2 2) - \frac{i\pi^2}{384} (-13 + 24 \ln 2) \\
q_{22}^{\pm} &= \pm \frac{\pi}{512} (4\pi^2 - 3(-1 + 8 \ln 2 + 64 \ln^2 2)) - \frac{3i\pi^2}{512} (1 + 8 \ln 2) \\
q_{23}^{\pm} &= \pm \frac{\pi}{512} (63 - 48 \ln 2(3 - 2 \ln 2 + 4 \ln^2 2) + 4\pi^2(-3 + 4 \ln 2) - 84\zeta(3)) + \\
&\quad + \frac{i\pi^2}{256} (\pi^2 - 3(7 - 12 \ln 2 + 8 \ln^2 2))
\end{aligned} \tag{A.80}$$

and

$$\begin{aligned}
Q_{21}^{\pm} &= -\frac{i\pi}{1152} (-71 + 3\pi^2 + 12(13 - 12 \ln 2) \ln 2) \\
Q_{22}^{\pm} &= \frac{i\pi}{512} (3 + \pi^2 - 12 \ln 2(1 + 4 \ln 2)) \\
Q_{23}^{\pm} &= \pm \frac{\pi^2}{512} (\pi^2 - 3(-1 + 4 \ln 2 + 16 \ln^2 2)) - \\
&\quad - \frac{i\pi}{4608} (283 - 1728 \ln^3 2 + 6\pi^2(2 + 42 \ln 2) + 96 \ln 2(-7 + 3 \ln 2) - 756\zeta(3))
\end{aligned} \tag{A.81}$$

where  $\zeta(3)$  represents the Riemann  $\zeta$ -function.

#### 4 . Additional Relations

$$\begin{aligned}
w_{12}(\Theta_{12}^- - \Theta_{21}^-) &= w_{12}(\Theta_{12}^+ - \Theta_{21}^+) = 0 \\
S_{22}^-(f_0, g_0) - S_{22}^+(f_0, g_0) &= -\frac{3i\pi^3}{4}, \quad U_{22}^-(f_2) - U_{22}^+(f_2) = 4i\pi^3 \\
S_{22}^-(f_4, g_4) - S_{22}^+(f_4, g_4) &= \frac{10i\pi^3}{3}, \quad S_{22}^-(\hat{f}_4, \hat{g}_4) - S_{22}^+(\hat{f}_4, \hat{g}_4) = \frac{17i\pi^3}{6} \\
\mathcal{S}_{22}^- - \mathcal{S}_{22}^+ &= \frac{i\pi^3}{12} \left( -7 + 12 \ln 2 + \frac{9\pi^2}{32} \right).
\end{aligned} \tag{A.82}$$

Finally, we see that divergent terms remain in the following expression

$$\begin{aligned}
w_{12}(\Delta_{12}^+ - \Delta_{12}^-)(\Delta_{22}^- + \pi^2 \Delta_{12}^-) &= \\
&= \left( \frac{1}{4}(11 - 12 \ln 2) - \frac{3}{4} \lim_{z \rightarrow \infty} \ln z \right) \times \frac{\pi^2}{8} \left( (11 - 12 \ln 2) + 3 \lim_{z \rightarrow 1} \ln(z - 1) \right).
\end{aligned} \tag{A.83}$$

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